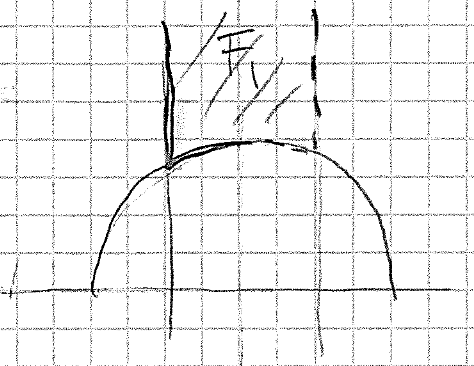
Remark ① Let $\overline{F_1} = \text{int } F \cup \left\{ \frac{-1}{2} + it \mid t \in \left[\frac{\sqrt{3}}{2}, \infty\right) \right\}$

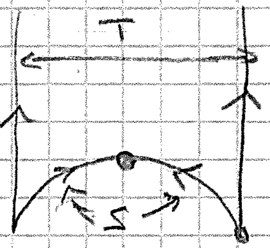
$\cup \left\{ e^{i\theta} \mid \frac{\pi}{2} \leq \theta < 2\frac{\pi}{3} \right\}$



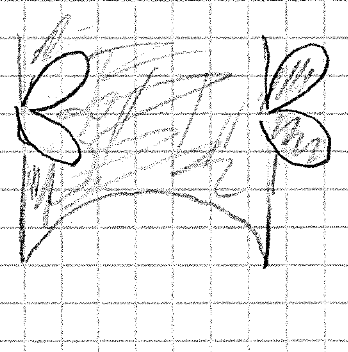
Then every point $z \in \mathbb{H}$ is Γ-equivalent to a single point in $\overline{F_1}$

$\overline{F_1}$ is called strict fundamental domain

② Note S and T identifies the 2 boundary lines and generate Γ



③ Note that any image γF of F is also a fund. domain. Also you can take a piece A of F and translate it by an elt of γ then $(F \setminus A) \cup \gamma A$ is also a fund domain



Important s/gps of Γ

There are some important s/gps of Γ
In particular the so called congruence s/gps.
let $N \in \mathbb{N}$, $N > 0$

Defn. $\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

is called the principal congruence s/gp of level N.

$\Gamma(N)$ is the kernel of the homomorphism

$$\Gamma = \Gamma(1) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$$

Hence $\Gamma(N) \trianglelefteq \Gamma$.
Since $SL_2(\mathbb{Z}/N\mathbb{Z})$ is a finite group

$$[\Gamma : \Gamma(N)] < \infty.$$

Defn A subgroup of Γ is called a congruence subgroup if it contains $\Gamma(N)$ for some N.

Rk A cong. s/gp of level N also has level M for any M which is a multiple of N since $\Gamma(M) \subset \Gamma(N)$ if $M = N\ell$

eg $\Gamma(4) \subset \Gamma(2) \subset \Gamma(1) = \Gamma$.

Other than the principal cong. s/g $\Gamma(N)$ we have the following important s/gps

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0\,(N) \right\} \supset \Gamma(N)$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv 1\,(N) \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \begin{matrix} a \equiv 1\,(N) \\ c \equiv 0\,(N) \end{matrix} \right\} \supset \Gamma(N)$$

<u>Exercise</u>: If $\Gamma' < \Gamma$ is a s/gp of finite index, ie $[\Gamma : \Gamma'] < \infty$ and

$\Gamma = \bigcup_{i=1} \Gamma' \alpha_i$ and F is a fund. domain of Γ then $F' = \bigcup_{i=1} \alpha_i F$ is a fund domain for Γ'.

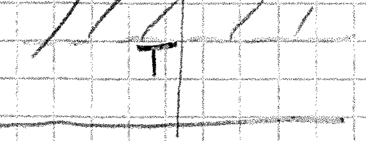
<u>Rk.</u> It is possible to define a topology on $\Gamma \backslash \overline{\mathbb{H}}$ (See Diamond-Schurman Chapter 2) (esp Fig 2.7.)

(1) We start with a topology on $\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$

Namely if $z_0 \in \mathbb{H}$, we take as fund system of nbhds of z_0, the usual open sets in $\mathbb{H}$ containing z_0

If $z_0 = i\infty$ we take the sets

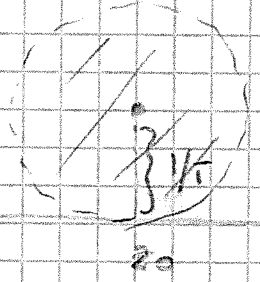
$$\mathbb{H}_T^{i\infty} := \{ z \in \mathbb{H} \mid \operatorname{Im} z > T \} \cup \{ i\infty \}$$



$\forall T$ as nbhds of $i\infty$

If $z_0 \in$ ② we take the sets

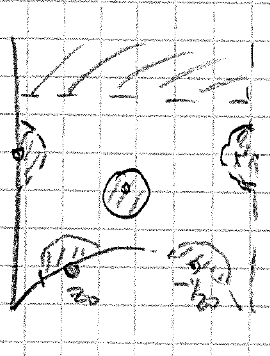
$$\mathbb{H}_r^{z_0} := \left\{ z \in \mathbb{H} \;\middle|\; \left| z - z_0 - \frac{i}{r} \right| < \frac{1}{r} \right\} \cup \{z_0\}.$$



② We then endow $\Gamma\backslash\overline{\mathbb{H}}$ with the quotient topology, i.e. if $\pi : \overline{\mathbb{H}} \longrightarrow \Gamma\backslash\overline{\mathbb{H}}$ is the canonical surjection, a subset U of $\Gamma\backslash\overline{\mathbb{H}}$ is open iff $p^{-1}(U)$ is an open set of $\overline{\mathbb{H}}$

③ This gives the following topology on the complete system of representatives $= F_1$

For $z_0 \in \operatorname{int} F$, we take as a fund system of nbhds the usual open sets containing z_0



If $z_0 = i\infty$ we take the set

$$B_r = \{ z \in F,\ \operatorname{Im} z > r \} \cup \{ i\infty \}$$

If $\operatorname{Re} z_0 = -\frac{1}{2}$ we take the set

$$\{ z \in F \mid |z - z_0| < \varepsilon \text{ or } |z - z_0 - 1| < \varepsilon \}$$

If $|z_0| = 1$ we take $\{ z \in F \mid |z - z_0| < \varepsilon$ or $\left| z + \frac{1}{z_0} \right| < \varepsilon$

One can then show that $\Gamma\backslash\overline{\mathbb{H}}$ is a compact topological space

In fact $\Gamma\backslash\mathbb{H}$ has a much richer structure

It is a compact analytic manifold.

The maps $z \longmapsto \dfrac{z - z_0}{z - \overline{z_0}}$ for $z_0 \neq i, \rho, i\infty$

$$z \longmapsto \left(\frac{z - i}{z + i}\right)^2 \quad \text{for } z = i$$

$$z \longmapsto \left(\frac{z - \rho}{z - \overline{\rho}}\right)^3 \quad \text{for } z = \rho$$

$$z \longmapsto e^{2\pi i z} \quad \text{for } z = i\infty$$

give $\Gamma\backslash\mathbb{H}$ the structure of a complex analytic manifold of dimension 1.

ie These maps are homeomorphisms from fund. system of nbhds of z_0 onto a fund. syst. of nbhds of 0 in $\mathbb{C}$

and the transition morphisms are holomorphic.

These functions from $\Gamma\backslash\mathbb{H}$ to $\mathbb{C}$ are called local uniformizers at $z_0, i, \rho, i\infty$ resp.)

We'll mainly use the uniformizer at $i\infty$, $e^{2\pi i z} =: q$

Note the nbhds of i are only half "discs" whence the square, the nbhds of ρ are only a third of discs hence the cube. 2, 3 correspond to the cardinality of the isotropy groups in $PSL_2(\mathbb{Z})$.

§2 Basic definitions and properties of Modular forms

We start with a definition

Defn Let $k \in \mathbb{Z}$, $g \in SL_2(\mathbb{R})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$f : \mathbb{H} \to \mathbb{C}$. We define the <u>weight k</u> <u>slash operator</u> on functions by

$$(f|_k g)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

The group $G = SL_2(\mathbb{R})$ acts on functions via the slash operator

$$f|_k g_1 |_k g_2 = f| g_1 g_2$$

This is a consequence of the automorphy condition for $j(g, z) = cz + d$

Namely

$$j(g_1 g_2, z) = j(g_1, g_2 z) j(g_2, z)$$

For wt $k = 0$, this action means

$$(f| g)(z) = f\left(\frac{az+b}{cz+d}\right)$$

(If the wt is clear we will drop it and write $f|g$ instead of $f|_k g$

For wt $k = 2$ since $\dfrac{d(g\tau)}{d\tau} = \dfrac{1}{(c\tau + d)^2}$

$$(f|_2 g)(\tau) = f(g\tau) \frac{d(g\tau)}{d\tau}$$

Defn Let $k \in \mathbb{Z}$. A meromorphic function $f: \mathbb{H} \to \mathbb{C}$ is called weakly modular of weight k for $\Gamma = SL_2(\mathbb{Z})$ if

$$(f|_k \gamma)(\tau) = f(\tau) \quad \forall \gamma \in \Gamma. \quad (*)$$

Rk ① Since $f|_k \gamma_1\gamma_2 = f|_k \gamma_1|_k \gamma_2$ and $\Gamma = \langle S, T \rangle$ to verify $(*)$ it is enough to verify it for S, and T

ie $(*) \equiv \begin{cases} (f|_k S)(\tau) = f(\tau) \\ (f|_k T)(\tau) = f(\tau) \end{cases}$

ie. $\begin{cases} f(-\frac{1}{\tau}) = \tau^k f(\tau) \\ f(\tau+1) = f(\tau) \end{cases}$

② weakly modular of wt 0 means f is invariant under the action of Γ

ie $f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \quad \forall \gamma \in \Gamma$

③ weakly modular of wt 2 is also natural in terms of invariant differentials

$$(f|_2 \gamma)(\tau) = \frac{d(\gamma\tau)}{d\tau} f(\gamma\tau)$$ since $\frac{d(\gamma\tau)}{d\tau} = \frac{1}{(c\tau+d)^2}$

~~Hence~~ if $(f|_2 \gamma)(\tau) = f(\tau)$ then

$$f(\gamma\tau)\, d(\gamma\tau) = f(\tau)\, d\tau.$$

④ If $\sigma = -I$ $\left(f|_k(-I)\right)(\tau) = f(\tau)$

$$\Rightarrow f(\tau) = (-1)^k f(\tau)$$

Hence if k is odd then $f \equiv 0$.

Non-zero odd weight modular functions exist for some subgroups $\Gamma' < \Gamma$ with $-I \notin \Gamma'$!

<u>Defn</u> A function $f : \mathbb{H} \to \mathbb{C}$ is called a <u>modular form of wt k</u> for Γ if

① f is holomorphic on $\mathbb{H}$

② f is weakly modular of wt k

ie $f|_k \sigma = f \quad \forall \sigma \in \Gamma$

③ f is "holomorphic at ∞".

To define holomorphic at ∞ let

$$D' = \{ q \in \mathbb{C} \mid 0 < |q| < 1 \}$$ be the punctured disc.

Consider the function

$$q : \mathbb{H} \longrightarrow D'$$
$$\tau \longmapsto e^{2\pi i \tau} = q$$

The function $\tilde{f} : D' \longrightarrow \mathbb{C}$

$$q \longmapsto f\left(\frac{\log q}{2\pi i}\right)$$

is well defined, ie does not depend on the choice of logarithm. This is because choosing a different branch of the logarithm changes the value by a multiple of $2\pi i$

But since f is $\mathbb{Z}$-periodic

$\tilde{f}(q)$ does not depend on the choice of the logarithm, and $f(\tau) = \tilde{f}(e^{2\pi i \tau})$

If f is holom on $\mathbb{H}$ then $\tilde{f}$ is holom in D' and has a Laurent expansion

$$\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n \quad \text{for } q \in D'$$

$$a_n = \frac{1}{2\pi i} \int_{|\eta| = r} \frac{\tilde{f}(\eta)}{\eta^{n+1}} \, d\eta$$

This means $f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau}$

Defn We say

f is holomorphic at ∞ if $\tilde{f}$ extends holomorphically to the point $q = 0$

ie. The Laurent series for $\tilde{f}$ does not have negative powers of q

(Similarly meromorphic at ∞ means the Laurent series for $\tilde{f}$ has only finitely many negative powers of q).

The numbers a_n are called Fourier coefs of f

Rk ① If we want to show that
$f : \mathbb{H} \to \mathbb{C}$ with $f(\tau+1) = f(\tau)$
is holom. at ∞, we do not have to compute its Fourier expansion but just show that $\lim_{\tau \to i\infty} f(\tau)$ exists
or even just bounded as $\mathrm{Im}\,\tau \to \infty$

$f(\tau+1) = f(\tau)$ $\Longrightarrow$ $f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n\tau}$
+ f holom on $\mathbb{H}$

$$\left|e^{2\pi i n(\tau)}\right| = \left|e^{2\pi i n x}\right| \left|e^{-2\pi n y}\right| = e^{-2\pi n y}$$

as $y \to \infty$ $e^{-2\pi n y} \longrightarrow 0$ if $n > 0$
$\longrightarrow$ constant if $n = 0$
$\longrightarrow \infty$ if $n < 0$

f holom at $\infty \Leftrightarrow a_n = 0$ for $n < 0$
In this case f is bounded as $\mathrm{Im}\,\tau \to \infty$
and $\lim_{\tau \to i\infty} f(\tau) = a_0$.

Defn $\mathcal{M}_k(\Gamma) := \{ f : \mathbb{H} \to \mathbb{C} \mid f$ is a mod. form of wt k for $\Gamma \}$

The space of modular forms of wt k
$\mathcal{M}_k(\Gamma)$ is a vector space and we'll see soon that $\dim \mathcal{M}_k(\Gamma) < \infty$

$f \in \mathcal{M}_k(\Gamma)$ is called <u>a cusp form</u> if $a_0 = 0$
and we denote the space of cusp forms by $S_k(\Gamma)$

Is there a function in $M_k(\Gamma)$? We saw $M_k(\Gamma)=\emptyset$ if $k=$ odd

Defn Eisenstein series

For $k>2$ even, $k\in\mathbb{N}$

let $$G_k(\tau) := \sum_{\substack{m,n\\ \in\mathbb{Z}}}{}' \frac{1}{(m\tau+n)^k}$$

$\sum_{m,n}'$ means $\sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\neq(0,0)}}$

Prop 2.1 The sum $\sum_{m,n}' \frac{1}{(m\tau+n)^k}$ converges absolutely and uniformly on compact ssets of $\mathbb{H}$, and hence defines a holomorphic function on $\mathbb{H}$.

Cor 2.2 $G_k(\tau)$ is a mod. form of wt k.

Proof of Cor. Since the sum conv. abs. and uniformly on compacta its terms can be rearranged at will. To check modularity we need to check $G(\tau+1)=G(\tau)$ and
$$G\left(-\frac{1}{\tau}\right)=\tau^k G(\tau)$$

$$G_k(\tau+1)=\sum_{m,n}{}' \frac{1}{(m\tau+m+n)^k} = G_k(\tau)$$

since as $\{(m,n)\}$ runs over $\mathbb{Z}^2\setminus\{0,0\}$ so does $\{(m,m+n)\}$.

Similarly $$G_k\left(-\frac{1}{\tau}\right) = \sum_{m,n}{}' \frac{1}{\left(m\left(-\frac{1}{\tau}\right)+n\right)^k}$$

$$= \tau^k \sum_{m,n} \frac{1}{(n\tau - m)^k} = G_k(\tau)$$

(In fact one could even check directly that

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, G(\tau) \quad \text{since}$$

As $\{(m,n)\}$ runs over $\mathbb{Z}^2 \setminus \{(0,0)\}$

so does $\{ma+nc, (mb+nd)\} = \left\{(m\ n)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\}$.)

To see G is holom at ∞ we

look at $$\lim_{\tau\to i\infty} \sum_{m,n}{}' \frac{1}{(m\tau+n)^k} =$$

$$\lim_{\tau\to i\infty}\left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0\\ m=0}} \frac{1}{n^k} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} (m\tau+n)^{-k}\right)$$

$$= 2\zeta(k) + \lim_{\mathrm{Im}\,\tau\to\infty} 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}} \cancelto{0}{(m\tau+n)^{-k}}\right)$$

Hence $$\lim_{\tau\to i\infty} G_k(\tau) = 2\zeta(k) < \infty. \quad \text{for } k>2$$

Next we prove Prop 2.1

We first need a lemma.

Lemma 2.3 Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Then the series

$$\sum_{\substack{\omega \in L \\ \omega \neq 0}} |\omega|^{-s} \text{ converges for } \mathrm{Re}\, s > 2.$$

Proof of lemma Since $\left||\omega|^{-s}\right| = |\omega|^{-\mathrm{Re}\, s}$

we may assume s is real.

Set $\tau = \omega_1/\omega_2 = x + iy$, $y \neq 0$.

$$|m\omega_1 + n\omega_2|^2 = |\omega_2|^2 |m\tau + n|^2$$

$$\geq A(m^2 + n^2)$$

with $A = |\omega_2|^2 \dfrac{y^2}{x^2 + y^2 + 1} > 0$

Since for any integer m, n we have

$$\frac{y^2}{x^2+y^2+1} \leq \frac{|m\tau + n|^2}{m^2+n^2}.$$

This inequality is equivalent to

$\left((mx+n)^2 + m^2y^2\right)(x^2+y^2+1) - (m^2+n^2)y^2 \geq 0$. But

LHS $= (mx+n)^2 + \left(m(x^2+y^2) + nx\right)^2$ Check this!

Side remark

We don't need this by similarly one can show $\frac{|m\tau+n|^2}{m^2+n^2} \le x^2+y^2+1$

which is equivalent to

$$\underbrace{(x^2+y^2+1)(m^2+n^2) - \left((mx+n)^2+m^2y^2\right)}_{} \ge 0$$

$$= (nx-m)^2 + n^2y^2$$

Thus we have $\sum'_{w \in L} |w|^{-s} \le A^{-s/2} \sum'_{m,n} (m^2+n^2)^{-s/2}$

But now the sum $\sum'_{m,n} \frac{1}{(m^2+n^2)^\alpha}$ converges iff $\alpha > 1$

To see this

note $I = \int_{x^2+y^2 \ge 1} \frac{dx\,dy}{(x^2+y^2)^\alpha}$ converges $\Leftrightarrow$ $\alpha > 1$

The convergence of the integral can be seen using Polar coordinates

$$\int_0^{2\pi}\int_1^\infty \frac{r\,dr\,d\theta}{r^{2\alpha}} = 2\pi \int_1^\infty \frac{dr}{r^{2\alpha-1}}$$

Int conv $\Leftrightarrow$ $2\alpha - 1 > 1$ $\Leftrightarrow$ $\alpha > 1$

Thus $\sum_{w \in L} |w|^{-s}$ converges if $s/2 > 1$

ie $s > 2$.